# Topology of positively curved 8-dimensional manifolds with symmetry

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**Abstract**

In this paper we show that a simply connected 8-dimensional manifold $M$ of positive sectional curvature and symmetry rank $\geq 2$ resembles a rank one symmetric space in several ways. For example, the Euler characteristic of $M$ is equal to the Euler characteristic of $S^8$, $\mathbb{H}P^2$ or $\mathbb{C}P^4$. And if $M$ is rationally elliptic then $M$ is rationally isomorphic to a rank one symmetric space. For torsion-free manifolds we derive a much stronger classification. We also study the bordism type of 8-dimensional manifolds of positive sectional curvature and symmetry rank $\geq 2$. As an illustration we apply our results to various families of 8-manifolds.

## 1 Introduction

In this paper we study the topology of positively curved 8-dimensional manifolds with symmetry rank $\geq 2$. Here a Riemannian manifold $M$ is said to have positive curvature if the sectional curvature of all its tangent planes is positive. The symmetry rank of $M$ is defined as the rank of its isometry group. Throughout this paper all manifolds are assumed to be closed, i.e. compact without boundary.

At present only few manifolds are known to admit a Riemannian metric of positive curvature. Besides the examples of Eschenburg and Bazaikin which are biquotients of dimension 6, 7 or 13 all other simply connected positively curved examples[1] are homogeneous, i.e. admit a metric of positive curvature with transitive isometry group (the latter were classified by Berger, Wallach, Aloff and Bérard Bergery). Moreover in dimension $> 24$ all known examples are symmetric of rank one.

Classifications of various strength have been obtained for positively curved manifolds with large symmetry (cf. Section 4 of the survey by Wilking [60]). Among the measures of "largeness" we shall focus on the symmetry rank.

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[1]Recently Petersen-Wilhelm [47], Grove-Verdiani-Ziller [24] and Dearricott announced the discovery of new 7-dimensional examples of positive curvature.

Grove and Searle [23] showed that the symmetry rank of a positively curved simply connected $n$-dimensional manifold $M$ is $\leq [\frac{n+1}{2}]$ and equality occurs if and only if $M$ is diffeomorphic to a sphere or a complex projective space. Wilking [59] proved that $M$ is homeomorphic to a sphere or a quaternionic projective space or $M$ is homotopically equivalent to a complex projective space provided the symmetry rank of $M$ is $\geq (\frac{n}{4}+1)$ and $n \geq 10$ (it follows from [11] that in this classification "homotopically equivalent" can be strengthened to "tangentially equivalent"). Building on [59] Fang and Rong [16] showed that $M$ is homeomorphic to a sphere, a quaternionic projective space or a complex projective space if the symmetry rank is $\geq [\frac{n-1}{2}]$ and $n \geq 8$.

In dimension eight the rank one symmetric spaces $S^8$, $\mathbb{H}P^2$ and $\mathbb{C}P^4$ are the only known simply connected positively curved examples. In this dimension the just mentioned work of Grove-Searle and Fang-Rong says that a positively curved simply connected manifold $M$ is diffeomorphic to $S^8$ or $\mathbb{C}P^4$ if the symmetry rank of $M$ is $\geq 4$ and homeomorphic to $S^8$, $\mathbb{H}P^2$ or $\mathbb{C}P^4$ if the symmetry rank is $\geq 3$.

The main purpose of this paper is to give some information on the topology of positively curved 8-dimensional manifolds with symmetry rank $\geq 2$. Our first result concerns the Euler characteristic.

**Theorem 1.1.** *Let $M$ be a simply connected 8-dimensional manifold. If $M$ admits a metric of positive curvature and symmetry rank $\geq 2$ then the Euler characteristic of $M$ is equal to the Euler characteristic of $S^8$, $\mathbb{H}P^2$ or $\mathbb{C}P^4$, i.e. $\chi(M) = 2, 3$ or 5.*

This information on the Euler characteristic leads to a rather strong classification if one assumes in addition that the manifold is rationally elliptic. Recall that a closed simply connected manifold $M$ is *rationally elliptic* if its rational homotopy $\pi_*(M) \otimes \mathbb{Q}$ is of finite rank. A conjecture attributed to Bott asserts that any non-negatively curved manifold is rationally elliptic (cf. [22, p. 172]).

**Theorem 1.2.** *Let $M$ be a simply connected positively curved 8-dimensional manifold of symmetry rank $\geq 2$.*

1. *If $M$ is rationally elliptic then $M$ has the rational cohomology ring and the rational homotopy type of a rank one symmetric space, i.e. of $S^8$, $\mathbb{H}P^2$ or $\mathbb{C}P^4$.*
2. *If $M$ is rationally elliptic and $H^*(M;\mathbb{Z})$ is torsion-free then $M$ is homeomorphic to $S^8$, diffeomorphic to $\mathbb{H}P^2$ or tangentially equivalent to $\mathbb{C}P^4$.*

If one drops the assumption on rational ellipticity and weakens the assumption on the symmetry rank one can still prove the following bound on the Euler characteristic.

**Theorem 1.3.** *Let $M$ be a simply connected positively curved manifold of even dimension $\leq 8$. Assume $S^1$ acts smoothly on $M$. If some $\sigma \in S^1$ acts isometrically and non-trivially on $M$ then $\chi(M) \geq 2$.*

This fits well with the Hopf conjecture on the positivity of the Euler characteristic of even dimensional positively curved manifolds.

To put our results in perspective we briefly recall what is known about positively curved manifolds in low dimensions. Next to surfaces manifolds of positive curvature are only classified in dimension 3 by the work of Hamilton and Perelman [27, 44, 45, 46]. In higher dimensions the only known obstructions to positive curvature for simply connected manifolds are given by Gromov's Betti number theorem [21] and the obstructions to positive scalar curvature (e.g. the $\alpha$-invariant of Lichnerowicz-Hitchin and the obstructions in dimension 4 coming from Seiberg-Witten theory). In particular, the Hopf problem which asks whether $S^2 \times S^2$ admits a metric of positive curvature is still open.

The study of low dimensional positively curved manifolds with positive symmetry rank began with the work of Hsiang and Kleiner [32] on 4-dimensional manifolds. Their main result asserts hat the Euler characteristic of a simply connected positively curved 4-dimensional manifold $M$ with positive symmetry rank is $\leq 3$. Using Freedman's work [19] Hsiang and Kleiner conclude that $M$ is homeomorphic[2] to $S^4$ or $\mathbb{C}P^2$. Rong [50] showed that a simply connected positively curved 5-dimensional manifold with symmetry rank 2 is diffeomorphic to $S^5$. In dimension 6 (resp. 7) there are examples of symmetry rank 2 (resp. 3) which are not homotopically equivalent to a rank one symmetric space [1, 13]. This indicates that in these dimensions a classification below the maximal symmetry rank is more complicated.

Theorem 1.1 and Theorem 1.3 imply that for many non-negatively curved manifolds any metric of positive curvature must be quite non-symmetric. For example if $M$ is of even dimension $\leq 8$ and has Euler characteristic $< 2$ (e.g. $M$ is a product of two simply connected odd dimensional spheres or $M$ is a simply connected Lie group) then it follows from Theorem 1.3 that for any positively curved metric $g$ on $M$ the only isometry of $(M, g)$ sitting in a compact connected Lie subgroup of the diffeomorphism group is the identity. As a further illustration of our results we consider the following classes of manifolds.

**(i) Product manifolds:** Let $M$ be a simply connected product manifold $M = N_1 \times N_2$ of dimension 8 ($\dim N_i > 0$). It is straightforward to see that the Euler characteristic of $M$ is $\neq 2, 3$ or 5. By Theorem 1.1 $M$ does not admit a metric of positive curvature with symmetry rank $\geq 2$. In particular, the product of two simply connected non-negatively curved manifolds $N_1$ and $N_2$ (e.g. $S^4 \times S^4$) does not admit a metric of positive curvature and symmetry rank $\geq 2$. It is interesting to compare this with the work of Hsiang-Kleiner [32] which implies that $S^2 \times S^2$ does not admit a metric of positive curvature and symmetry rank $\geq 1$.

**(ii) Connected sum of rank one symmetric spaces:** Cheeger [9] has shown that $\mathbb{C}P^4 \sharp \pm \mathbb{C}P^4$, $\mathbb{C}P^4 \sharp \pm \mathbb{H}P^2$ and $\mathbb{H}P^2 \sharp \pm \mathbb{H}P^2$ admit a metric of non-negative curvature. The Euler characteristic of these manifolds is 8, 6 and 4, respectively. By Theorem 1.1 none of them admits a metric of positive curvature and symmetry rank $\geq 2$.

[2] According to the recent preprint of Kim [36] $M$ is diffeomorphic to $S^4$ or $\mathbb{C}P^2$.

**(iii) Cohomomogeneity one manifolds:** In [25] Grove and Ziller constructed invariant metrics of non-negative curvature on cohomogeneity one manifolds with codimension two singular orbits. Using this construction they exhibited metrics of non-negative curvature on certain infinite families of simply connected manifolds which fibre over $S^4$, $\mathbb{C}P^2$, $S^2 \times S^2$ or $\mathbb{C}P^2 \sharp \pm \mathbb{C}P^2$ [25, 26] (see also the survey [61] of Ziller). In dimension 8 the Euler characteristic of all these manifolds turns out to be $\neq 2, 3, 5$. Again, by Theorem 1.1 none of them admits a metric of positive curvature and symmetry rank $\geq 2$.

**(iv) Biquotients:** Another interesting class of manifolds known to admit metrics of non-negative curvature are biquotients. A biquotient of a compact Lie group $G$ is the quotient of a homogeneous space $G/H$ by a free action of a subgroup $K$ of $G$, where the $K$-action is induced from the left $G$-action on $G/H$. Note that any homogeneous space can be described as a biquotient by taking one of the factors to be trivial. If $G$ is equipped with a bi-invariant metric the biquotient $M = K\backslash G/H$ inherits a metric of non-negative curvature, a consequence of O'Neill's formula for Riemannian submersions. As pointed out by Eschenburg [15] a manifold $M$ is a biquotient if and only if $M$ is the quotient of a compact Lie group $G$ by a free action of a compact Lie group $L$, where the action of $L$ on $G$ is given by a homomorphism $L \to G \times G$ together with the two-sided action of $G \times G$ on $G$ given by $(g_1, g_2)(g) := g_1 \cdot g \cdot g_2^{-1}$.

The topology of biquotients has been investigated by Eschenburg [15], Singhof [51], Kapovich [34], Kapovich-Ziller [35] and Totaro [56]. In [35] Kapovich and Ziller classified biquotients with singly generated rational cohomology. Combining their classification with the first part of Theorem 1.2 gives

**Corollary 1.4.** *A simply connected* 8*-dimensional biquotient of positive curvature and symmetry rank* $\geq 2$ *is diffeomorphic to* $S^8$, $\mathbb{C}P^4$, $\mathbb{H}P^2$ *or* $G_2/SO(4)$.

In view of Theorem 1.1 and Theorem 1.3 the examples above contain plenty examples of simply connected non-negatively curved manifolds with positive Ricci curvature for which the metric cannot be deformed to a metric of positive curvature via a symmetry preserving process such as the Ricci flow.

The paper is structured as follows. In the next section we recall basic geometric and topological properties of positively curved manifolds with symmetry. In Section 3 we prove the statements on the Euler characteristic. In Section 4 we prove our classification result for rationally elliptic manifolds (see Theorem 1.2) and the corollary for biquotients (see Corollary 1.4). In the final section we study the bordism type of positively curved 8-dimensional manifolds with symmetry.

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A particular step in the proof of Theorem 1.2 relies on a result of Masuda [41] about $\mathbb{Z}/2\mathbb{Z}$-cohomology $\mathbb{C}P^4$'s. I am greatful to Mikiya Masuda for sharing his insight with me.

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# 2 Tools from geometry and topology

On the topological side the proofs rely on arguments from equivariant index theory (see Theorem 2.4 below) and the cohomological structure of fixed point sets of smooth actions on cohomology spheres and cohomology projective spaces (see Theorem 2.5 below).

On the geometric side the proofs rely on the work of Hsiang-Kleiner on positively curved 4-dimensional manifolds with symmetry (see Theorem 2.3 below), the fixed point theorems of Berger, Synge, Weinstein for isometries (cf. [37, Ch. II, Cor. 5.7] and [55, 58]) and the following two properties of totally geodesic submanifolds due to Frankel and Wilking, respectively, which we state for further reference.

**Intersection theorem ([17]):** Let $M$ be a connected positively curved manifold of dimension $n$ and let $N_1$ and $N_2$ be totally geodesic submanifolds of dimension $n_1$ and $n_2$, respectively. If $n_1 + n_2 \geq n$ then $N_1$ and $N_2$ intersect. ■

Here the dimension of a manifold is defined to be the maximal number occuring as the dimension of a connected component of the manifold. Similarly, the codimension of a submanifold $N$ of a connected manifold $M$ is defined to be the minimal number occuring as the codimension of a connected component of $N$ in $M$.

Building on the intersection theorem Frankel observed that the inclusion of a connected totally geodesic submanifold $N$ is 1-connected provided the codimension of $N$ in $M$ is at most half of the dimension of $M$ (cf. [18, p. 71]). Using a Morse theory argument Wilking proved the following far reaching generalization.

**Connectivity theorem ([59])**: Let $M$ be a connected positively curved manifold and let $N_1$ and $N_2$ be connected totally geodesic submanifolds of codimension $k_1$ and $k_2$, respectively.

1. Then the inclusion $N_i \hookrightarrow M$ is $(n - 2k_i + 1)$-connected.
2. Suppose $k_1 + k_2 \leq n$ and $k_1 \leq k_2$. Then the intersection $N_1 \cap N_2$ is a totally geodesic submanifold and the inclusion of $N_1 \cap N_2$ in $N_2$ is $(n - (k_1 + k_2))$-connected. ■

The connectivity theorem leads to strong restrictions on the topology of positively curved manifolds with large symmetry. In [59] Wilking used this property to classify positively curved manifolds of dimension $n \geq 10$ (resp. $n \geq 6000$) with symmetry rank $\geq \frac{n}{4}+1$ (resp. $\geq \frac{n}{6}+1$). For further reference we point out the following rather elementary consequences.

**Corollary 2.1.** *Let $M$ be a simply connected positively curved manifold of even dimension $n = 2m \geq 6$.*

1. *Suppose $M$ admits a totally geodesic connected submanifold $N$ of codimension 2. Then $N$ is simply connected. Moreover the integral cohomology of $M$ and $N$ is concentrated in even degrees and satisfies $H^{2i}(M;\mathbb{Z}) \cong H^{2j}(N;\mathbb{Z})$ for all $0 < 2i < n$ and all $0 < 2j < n-2$.*
2. *Suppose $M$ admits two different totally geodesic connected submanifolds $N_1$ and $N_2$ of codimension 2. Then $M$ is homeomorphic to $S^n$ or homotopy equivalent to $\mathbb{C}P^m$.*

**Proof:** The first part follows directly from the connectivity theorem (see [59]). For the convenience of the reader we recall the argument. We begin with a more general discussion.

Let $M$ be an oriented $n$-dimensional connected manifold and $N \overset{i}{\hookrightarrow} M$ an oriented connected submanifold of codimension $k$. Let $u \in H^k(M;\mathbb{Z})$ be the Poincaré dual of the fundamental class of $N$ in $M$. Then the cup product with $u$ is given by the composition of the maps

$$H^i(M;\mathbb{Z}) \xrightarrow{i^*} H^i(N;\mathbb{Z}) \xrightarrow{\cong} H_{n-k-i}(N;\mathbb{Z}) \xrightarrow{i_*} H_{n-k-i}(M;\mathbb{Z}) \xrightarrow{\cong} H^{i+k}(M;\mathbb{Z}),$$

where the second and forth map are the Poincaré isomorphism maps of $N$ and $M$, respectively (see for example [43, p. 137]). Now assume the inclusion $i : N \hookrightarrow M$ is $(n-k-l)$-connected. Then it is straightforward to check that the homomorphism

$$\cup u : H^i(M;\mathbb{Z}) \to H^{i+k}(M;\mathbb{Z}), \quad x \mapsto x \cup u,$$

is surjective for $l \leq i < n-k-l$ and injective for $l < i \leq n-k-l$ (for all of this see Lemma 2.2 in [59]).

In the first part of the corollary we have $k = 2$. By the connectivity theorem the inclusion $N \hookrightarrow M$ is $(n-3)$-connected, i.e. $l = 1$. Hence, the map $\cup u : H^i(M;\mathbb{Z}) \to H^{i+2}(M;\mathbb{Z})$ is surjective for $1 \leq i < n-3$ and injective for $1 < i \leq n-3$. Since $M$ is simply connected and $N \hookrightarrow M$ is at least 3-connected the first part follows.

Next assume $N_1$ and $N_2$ are different totally geodesic connected submanifolds of codimension 2. By the second part of the connectivity theorem $N := N_1 \cap N_2 \hookrightarrow N_1$ is a totally geodesic submanifold and the inclusion $N := N_1 \cap N_2 \hookrightarrow N_1$ is at least 2-connected.

If the codimension of $N$ in $N_1$ is two then the map $H^0(N_1;\mathbb{Z}) \to H^2(N_1;\mathbb{Z})$ (given by multiplication with the Poincaré dual of $N$ in $N_1$ for some fixed orientation of $N$) is surjective by the connectivity theorem. Hence, $b_2(M) = b_2(N_1) \leq 1$. Using the first part of the corollary we conclude that $M$ is an integral cohomology sphere or an integral cohomology $\mathbb{C}P^m$ if $b_2(M) = 0$ or 1, respectively. If $b_2(M) = 0$ then $M$ is actually homeomorphic to $S^n$ by the work of Smale [52]. If $b_2(M) = 1$ then $M$ is homotopy equivalent to $\mathbb{C}P^m$ since $M$ is simply connected.

Next assume the codimension of $N$ in $N_1$ is one. Using the connectivity theorem we see that the inclusion $N \hookrightarrow N_1$ is $(n-3)$-connected. Arguing along the lines above it follows that $M$ is homeomorphic to $S^n$. This completes the proof of the second part. ■

**Remark 2.2.** For 8-dimensional manifolds one can show that under the assumptions of the second part of Corollary 2.1 $M$ is homeomorphic to $S^8$ or *homeomorphic* to $\mathbb{C}P^4$. This follows from Sullivan's classification of homotopy complex projective spaces [54] (see the argument in [16] on page 85).

Another important geometric ingredient in our proofs is the classification of positively curved 4-dimensional manifolds with positive symmetry rank up to homeomorphism due to Hsiang and Kleiner.

**Theorem 2.3** ([32]). *Let $M$ be a positively curved simply connected 4-dimensional manifold with positive symmetry rank. Then the Euler characteristic of $M$ is 2 or 3 and, hence, $M$ is homeomorphic to $S^4$ or $\mathbb{C}P^2$ by Freedman's work.* ∎

In particular, $S^2 \times S^2$ does not admit a metric of positive curvature and positive symmetry rank. Note that Theorem 1.1 gives analogues restrictions for the Euler characteristic in dimension 8.

Among the topological tools used in the proofs are the classical Lefschetz fixed point formula for the Euler characteristic, the rigidity of the signature on oriented manifolds with $S^1$-action and its applications to involutions [30] as well as the Atiyah-Hirzebruch $\hat{A}$-vanishing theorem for $S^1$-actions on *Spin*-manifolds [3]. For further reference we shall recall these results in the following

**Theorem 2.4.** *Let $M$ be an oriented manifold with smooth non-trivial $S^1$-action, let $\sigma \in S^1$ be the element of order 2 and let $M^{S^1}$ (resp. $M^\sigma$) denote the fixed point manifold with respect to the $S^1$-action (resp. $\sigma$-action). Then:*

1. $\chi(M) = \chi(M^{S^1})$.
2. *the equivariant signature $sign_{S^1}(M)$ is constant as a character of $S^1$.*
3. *$sign(M) = sign(M^{S^1})$, where the orientation of each component of the fixed point manifold $M^{S^1}$ is chosen to be compatible with the complex structure of its normal bundle (induced by the $S^1$-action) and the orientation of $M$.*
4. *the signature of $M$ is equal to the signature of a transversal self-intersection $M^\sigma \circ M^\sigma$.*

*If $M$ is in addition a Spin-manifold, then:*

5. *the $\hat{A}$-genus vanishes.*
6. *the connected components of $M^\sigma$ are either all of codimension $\equiv 0 \bmod 4$ (even case) or all of codimension $\equiv 2 \bmod 4$ (odd case).*

**Proof:** The first statement is just a version of the classical Lefschetz fixed point theorem (see for example [37, Th. 5.5]). For the convenience of the reader we give a simple argument: For any prime $p$ choose a triangulation of $M$ which is equivariant with respect to the action of $\mathbb{Z}/p\mathbb{Z} \subset S^1$ on $M$. Then a counting argument shows $\chi(M) \equiv \chi(M^{\mathbb{Z}/p\mathbb{Z}}) \bmod p$. For $p$ large

enough this implies $\chi(M) = \chi(M^{S^1})$ (note that the proof also applies to non-orientable manifolds).

The second statement follows directly from the homotopy invariance of cohomology or from the Lefschetz fixed point formula of Atiyah-Bott-Segal-Singer (see [4, Theorem 6.12, p. 582]) as explained for example in [7, p. 142].

For the proof of the third statement consider the $S^1$-equivariant signature $sign_{S^1}(M) \in R(S^1) \cong \mathbb{Z}[\lambda, \lambda^{-1}]$ as a function in $\lambda \in \mathbb{C}$ and compute the limit $\lambda \to \infty$ using the Lefschetz fixed point formula (details can be found for example in [29, p. 68]).

The forth statement is a result of Hirzebruch (see [30] and [4, Prop. 6.15, p. 583]). Hirzebruch shows that the signature of a transversal self-intersection $M^\sigma \circ M^\sigma$ is equal to the equivariant signature $sign_{S^1}(M)$ evaluated at $\sigma$. Now the statement follows from the rigidity of the signature (see Part 2).

The fifth statement is the celebrated $\hat{A}$-vanishing theorem of Atiyah and Hirzebruch [3]. Using the Lefschetz fixed point formula the authors show that the $S^1$-equivariant $\hat{A}$-genus extends to a holomorphic function on $\mathbb{C}$ which vanishes at infinity. By a classical result of Liouville this function has to vanish identically.

For a proof of the last statement see [2, Prop. 8.46, p. 487]. ■

For latter reference we also point out certain properties of smooth actions on cohomology spheres and cohomology projective spaces.

**Theorem 2.5.** 1. *Suppose $M$ is a $\mathbb{Z}/2\mathbb{Z}$-cohomology sphere with $\mathbb{Z}/2\mathbb{Z}$-action. Then the $\mathbb{Z}/2\mathbb{Z}$-fixed point manifold is again a $\mathbb{Z}/2\mathbb{Z}$-cohomology sphere or empty.*

2. *Suppose $M$ is an integral cohomology sphere with $S^1$-action. Then the $S^1$-fixed point manifold is again an integral cohomology sphere or empty.*

3. *Suppose $M$ is a $\mathbb{Z}/2\mathbb{Z}$-cohomology complex projective space with $\mathbb{Z}/2\mathbb{Z}$-action such that the $\mathbb{Z}/2\mathbb{Z}$-action extends to an $S^1$-action. Then each component of the $\mathbb{Z}/2\mathbb{Z}$-fixed point manifold is again a $\mathbb{Z}/2\mathbb{Z}$-cohomology complex projective space.*

4. *The action of an involution on a $\mathbb{Z}/2\mathbb{Z}$-cohomology $\mathbb{C}P^2$ cannot have only isolated fixed points.* ■

**Proof:** The first two statements are well known applications of Smith theory (cf. [8, Ch. III, Th. 5.1, Th. 10.2]). The last two statements follow directly from the general theory on fixed point sets of actions on projective spaces (cf. [8, Ch. VII, Th. 3.1, Th. 3.3]). ■

# 3 Euler characteristic

In this section we prove the statements on the Euler characteristic given in the introduction. We begin with

**Theorem 3.1.** [Theorem 1.3] *Let $M$ be a simply connected positively curved manifold of even dimension $\leq 8$. Assume $S^1$ acts smoothly on $M$. If some $\sigma \in S^1$ acts isometrically and non-trivially on $M$ then $\chi(M) \geq 2$.*

**Proof:** Since $\sigma \in S^1$ acts non-trivially the dimension of $M$ is positive, i.e. $\dim M = 2, 4, 6$ or $8$. In dimension $\leq 4$ the theorem is true for purely topological reasons (Poincaré duality). So assume the dimension of $M$ is 6 or 8. Note that $M^\sigma$ is non-empty [58] and the connected components of the fixed point manifold $M^\sigma$ are totally geodesic submanifolds. Each is of even codimension since $\sigma$ preserves orientation.

If $M^\sigma$ contains a connected component of codimension 2 then, as pointed out in Corollary 2.1, the connectivity theorem implies that all odd Betti numbers of $M$ vanish. Hence, $\chi(M) \geq 2$ by Poincaré duality.

So assume codim $M^\sigma > 2$. Note that any connected component $F \subset M^\sigma$ is an $S^1$-invariant totally geodesic submanifold of even dimension $\leq 4$. If $F$ is not a point then $F$ inherits positive curvature from $M$. In this case $F$ or a two-fold cover of $F$ is simply connected by [55]. Hence, the Euler characteristic of any connected component of $M^\sigma$ is positive. From the Lefschetz fixed point formula for the Euler characteristic (see Theorem 2.4, Part 1) we get

$$\chi(M) = \chi(M^{S^1}) = \chi((M^\sigma)^{S^1}) = \chi(M^\sigma) = \sum_{F \subset M^\sigma} \chi(F) \geq 1.$$

Here equality holds if and only if $M^\sigma$ is connected and $\chi(M^\sigma) = 1$. If so, $M^\sigma$ must have the $\mathbb{Z}/2\mathbb{Z}$-cohomology of a point, of $\mathbb{R}P^2$ or of $\mathbb{R}P^4$. Note that the connected components of $M^{S^1} = (M^\sigma)^{S^1}$ are orientable submanifolds of even dimension $\leq 4$. This implies that the case $\chi(M) = 1$ can only happen if $M^{S^1}$ is a point (cf. [8, Ch. VII, Th. 3.1]).

However, a smooth $S^1$-action on a closed orientable manifold $M$ cannot have precisely one fixed point. To show this consider the Lefschetz fixed point formula [4] for the $S^1$-equivariant signature $sign_{S^1}(M)$. The local contribution for $sign_{S^1}(M)$ at an isolated $S^1$-fixed point extends to a meromorphic function on $\mathbb{C}$ which has at least one pole on the unit circle (see for example [7, p. 142]). Since $sign_{S^1}(M)$, being a character of $S^1$, has no poles on the unit circle the $S^1$-action cannot have precisely one fixed point (more generally this is true for any diffeomorphism of order $p^l$, $p$ an odd prime, as shown by Atiyah and Bott in [2, Th. 7.1]). Hence, $\chi(M) \geq 2$. ■

We remark that the proof simplifies drastically if $M$ has positive symmetry rank (see for example [49, Th. 2]). Note that by the result above any metric of positive curvature on $S^3 \times S^3$, $S^2 \times S^3 \times S^3$, $S^3 \times S^5$ or $SU(3)$ must be very non-symmetric.

In the remaining part of this section we restrict to positively curved simply connected 8-dimensional manifolds with symmetry rank $\geq 2$ and prove the statement on the Euler characteristic given in Theorem 1.1.

Let $T$ be a two-dimensional torus which acts isometrically and effectively on $M$, let $T_2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ denote the 2-torus in $T$ and let $\sigma \in T$ be a non-trivial involution, i.e. $\sigma \in T_2$, $\sigma \neq id$.

By Weinstein's fixed point theorem [58] the fixed point manifold $M^\sigma$ is non-empty. Each connected component $F$ of $M^\sigma$ is a totally geodesic $T$-invariant submanifold of $M$. Since $\sigma$ preserves orientation $F$ is of even codimension. By Berger's fixed point theorem (cf. [37, Ch. II, Cor. 5.7]) the torus $T$ acts with fixed point on $F$. For further reference we note the following

**Lemma 3.2.** *$F$ is orientable. If $\dim F \neq 6$ then $F$ is homeomorphic to $S^4$, $\mathbb{C}P^2$, $S^2$ or a point.*

**Proof:** If $\dim F = 6$ then $F$ is simply connected by the connectivity theorem and, hence, orientable.

Next suppose $\dim F = 4$ and $T$ acts trivially on $F$. In this case we can choose an $S^1$-subgroup of $T$ such that $M^{S^1}$ contains a 6-dimensional connected component $Y$ and $F$ is a $T$-fixed point component of $Y$. By [23, Th. 1.2], $M$, $Y$ and $F$ are diffeomorphic to spheres or complex projective spaces in this case.

Suppose $\dim F = 4$ and $T$ acts non-trivially on $F$. In this case we can find a subgroup $S^1 \subset T$ and a connected component $Y$ of $M^{S^1}$ such that $F \cap Y$ has positive dimension. If $\dim Y = 6$ then $M$ is diffeomorphic to $S^8$ or $\mathbb{C}P^4$ (cf. [23, Th. 1.2]) and $F$ is a a $\mathbb{Z}/2\mathbb{Z}$-cohomology sphere or a $\mathbb{Z}/2\mathbb{Z}$-cohomology complex projective space (cf. Theorem 2.5). Since the universal cover of $F$ is homeomorphic to $S^4$ or $\mathbb{C}P^2$ (cf. Theorem 2.3) we conclude that $F$ is simply connected and homeomorphic to $S^4$ or $\mathbb{C}P^2$. If $\dim Y = 4$ then $Y$ is homeomorphic to $S^4$ or $\mathbb{C}P^2$ by Theorem 2.3. It now follows from Theorem 2.5 that any connected component of $Y \cap F$ is a $\mathbb{Z}/2\mathbb{Z}$-cohomology sphere or $\mathbb{Z}/2\mathbb{Z}$-cohomology complex projective space. Since $F \cap Y$ has positive dimension $\chi(F) \geq 2$. Applying Theorem 2.3 again we conclude that $F$ is homeomorphic to $S^4$ or $\mathbb{C}P^2$.

Finally assume $\dim F = 2$. We choose an $S^1$-subgroup of $T$ which fixes $F$ pointwise. Let $Y$ be the connected component of $M^{S^1}$ which contains $F$. If $Y$ is 2-dimensional then $F = Y$ is orientable of positive curvature and, hence, diffeomorphic to $S^2$. If $Y$ is 6-dimensional then $M$ is diffeomorphic to $S^8$ or $\mathbb{C}P^4$ (cf. [23, Th. 1.2]) which implies $F \cong S^2$. If $Y$ is 4-dimensional then we may assume that $T$ acts non-trivially on $Y$ (otherwise we can replace $Y$ by a 6-dimensional connected component for some other $S^1$-subgroup). By Theorem 2.3 $Y$ is homeomorphic to $S^4$ or $\mathbb{C}P^2$. Using Theorem 2.5 we conclude that $F$ is diffeomorphic to $S^2$. ■

In the proof of Theorem 1.1 we will use the concept of the "type" of an involution at a fixed point component of the $T_2$-action. This concept is defined as follows. For each non-trivial involution $\sigma \in T_2$ and each connected component $X$ of $M^{T_2}$ we consider the dimension of the connected component $F$ of $M^\sigma$ containing $X$. For fixed $X$ this gives an unordered triple of dimensions which we call the *type* of $X$. Note that $F$ is orientable by Lemma 3.2. Since $T$ acts orientation preserving on $F$ any connected component $X$ of $M^{T_2}$ is a totally geodesic $T$-invariant submanifold of even dimension. By Berger's fixed point theorem (cf. [37, Ch. II, Cor. 5.7]) $X^T$ is non-empty.

We will use the following information on the type which can be easily verified by considering the isotropy representation at a $T$-fixed point in $X$.

**Lemma 3.3.** *1. The type of $X$ is $(6,6,4)$, $(6,4,2)$, $(6,2,0)$, $(4,4,4)$, $(4,4,0)$ or $(4,2,2)$.*

*2. $X$ is an isolated $T_2$-fixed point if and only if $X$ is of type $(6,2,0)$, $(4,4,0)$ or $(4,2,2)$.* ■

**Example 3.4.** *Let $M$ be the quaternionic plane*

$$\mathbb{H}P^2 = \{[q_0, q_1, q_2] \mid q_i \in \mathbb{H}, (q_0, q_1, q_2) \neq 0\},$$

*where $[q_0, q_1, q_2]$ denotes the orbit of $(q_0, q_1, q_2)$ with respect to the diagonal action of nonzero quaternions on $\mathbb{H}^3$ from the right. Consider the action of $T = S^1 \times S^1 = \{(\lambda, \mu) \mid \lambda, \mu \in S^1\} \subset \mathbb{C} \times \mathbb{C}$ on $M$ via*

$$(\lambda, \mu)([q_0, q_1, q_2]) := [\lambda \cdot \sqrt{\mu} \cdot q_0, \sqrt{\mu} \cdot q_1, \sqrt{\mu} \cdot q_2].$$

*Note that although the square root $\sqrt{\ }$ is only well defined up to sign the action is independent of this choice. Let $\sigma_1$ and $\sigma_2$ be the involutions in the first and second $S^1$-factor of $T$ and let $\sigma_3 := \sigma_1 \cdot \sigma_2$ denote the third non-trivial involution. Then we have the following fixed point manifolds*

$$\begin{array}{rcll} M^{\sigma_1} & = & \{[1,0,0]\} \cup \{[0, q_1, q_2] \mid q_i \in \mathbb{H},\ (q_1, q_2) \neq 0\} & \cong \ pt \cup S^4 \\ M^{\sigma_2} & = & \{[q_0, q_1, q_2] \mid q_i \in \mathbb{C},\ (q_0, q_1, q_2) \neq 0\} & \cong \ \mathbb{C}P^2 \\ M^{\sigma_3} & = & \{[j \cdot q_0, q_1, q_2] \mid q_i \in \mathbb{C},\ (q_0, q_1, q_2) \neq 0\} & \cong \ \mathbb{C}P^2 \\ M^{T_2} & = & \{[1,0,0]\} \cup \{[0, q_1, q_2] \mid q_i \in \mathbb{C},\ (q_1, q_2) \neq 0\} & \cong \ pt \cup S^2 \end{array}$$

*Hence, the type of $X = pt$ is $(4,4,0)$ and the type of $X = S^2$ is $(4,4,4)$.*

We now begin with the proof of Theorem 1.1 which is based on the following three lemmas.

**Lemma 3.5.** *If $\dim M^\sigma = 6$ for some involution $\sigma \in T$ then the Euler characteristic of $M$ is 2 or 5.*

**Proof:** Let $N \subset M^\sigma$ be the connected component of dimension 6. Note that all other connected components are isolated $\sigma$-fixed points by the intersection theorem. In view of Corollary 2.1 the odd Betti numbers of $M$ and $N$ vanish and the even Betti numbers satisfy $b_2(M) = b_4(M) = b_6(M) = b_2(N) = b_4(N)$. In particular, $\chi(M) - \chi(N) = b_2(M)$. By the Lefschetz fixed point formula for the Euler characteristic (see Theorem 2.4, Part 1) this difference is equal to the number of isolated $\sigma$-fixed points.

Suppose isolated $\sigma$-fixed points do occur (otherwise $b_2(M) = 0$ and $\chi(M) = 2$). Using Lemma 3.3, Part 2, we see that an isolated $\sigma$-fixed point is an isolated $T_2$-fixed point of type $(6,2,0)$ or $(4,4,0)$. If some isolated $\sigma$-fixed point is of type $(6,2,0)$ then $M$ contains a 6-dimensional fixed point manifold different from $N$. In this case $M$ is homeomorphic to $S^8$ or homotopy equivalent to $\mathbb{C}P^4$ (see Corollary 2.1, Part 2). In particular, $\chi(M) = 2$ or 5.

We now consider the remaining case. So suppose all isolated $\sigma$-fixed points are of type $(4,4,0)$. We fix a non-trivial involution $\sigma_1 \in T$ different from $\sigma$ and denote by $F_1$ the 4-dimensional connected component of $M^{\sigma_1}$

(note that $F_1$ is unique and has non-empty intersection with $N$ by the intersection theorem). Each isolated $\sigma$-fixed point (being of type $(4,4,0)$) is contained in $F_1$ and, hence, the number $d$ of isolated $\sigma$-fixed points is equal to $\chi(F_1) - \chi(F_1 \cap N)$. Since $T$ acts non-trivially on $F_1$ Theorem 2.3 tells us $\chi(F_1) \leq 3$. Using Lemma 3.3 we see that the connected components of $(F_1 \cap N)^{T_2}$ are necessarily 2-dimensional of type $(6,4,2)$. It follows from Lemma 3.2 that at least one of the connected components of $(F_1 \cap N)^{T_2}$ is diffeomorphic to $S^2$. Hence, $b_2(M) = d = \chi(F_1) - \chi(F_1 \cap N) \leq 1$ which in turn implies $\chi(M) = 2$ or $5$. ■

**Remark 3.6.** Under the assumptions of Lemma 3.5 $M$ is homeomorphic to $S^8$ or $\mathbb{C}P^4$. This follows from the proof above together with Smale's work on the high-dimensional Poincaré conjecture [52] and Sullivan's classification of homotopy complex projective spaces [54].

**Lemma 3.7.** *If* $\dim M^\sigma = 2$ *for some involution* $\sigma \in T$ *then* $\chi(M) = 2$.

**Proof:** We first note that the assumption on $\dim M^\sigma$ implies that the signature of $M$ vanishes by Theorem 2.4, Part 4. Since $\chi(M) \equiv sign(M) \bmod 2$ the Euler characteristic $\chi(M)$ is even.

By Lemma 3.5 we may assume that for every non-trivial involution $\tau$ of $T$ the dimension of the fixed point manifold $M^\tau$ is $\leq 4$. Using Lemma 3.3 we see that every $T_2$-fixed point component is an isolated fixed point of type $(4,4,0)$ or $(4,2,2)$.

Let $\sigma_1$ and $\sigma_2$ denote the non-trivial involutions different from $\sigma$. Without loss of generality we may assume that $\dim M^{\sigma_1} = 4$. Let $F_1$ denote the 4-dimensional connected component of $M^{\sigma_1}$. Since $T$ acts non-trivially on $F_1$ the universal cover of $F_1$ is homeomorphic to $S^4$ or $\mathbb{C}P^2$ by Theorem 2.3. Since all $T_2$-fixed points are isolated the involution $\sigma$ acts on $F_1$ with isolated fixed points. By Theorem 2.5 $F_1$ cannot be a cohomology $\mathbb{C}P^2$. Hence, $\chi(F_1) \leq 2$.

If $\dim M^{\sigma_2} < 4$ then any $T_2$-fixed point component is contained in $F_1$ and, hence, $\chi(M) = \chi(F_1) = 2$ by Theorem 1.3. So assume $M^{\sigma_2}$ contains a 4-dimensional connected component $F_2$. Arguing as above we see that $\chi(F_2) \leq 2$. Note that $F_1$ and $F_2$ intersect by the intersection theorem and that $F_1 \cap F_2$ consists of isolated $\sigma$-fixed points. Hence, $\chi(M) = \chi(F_1) + \chi(F_2) - \chi(F_1 \cap F_2) \leq 3$. Since $\chi(M)$ is even (as pointed out above) and $\geq 2$ (by Theorem 1.3) the lemma follows. ■

**Lemma 3.8.** *If* $\dim M^\sigma = 0$ *for some involution* $\sigma \in T$ *then* $\chi(M) = 2$.

**Proof:** The proof is very similar to the proof of Lemma 3.7. It follows from Theorem 2.4, Part 4, that the signature of $M$ vanishes. In particular, $\chi(M)$ is even.

Applying Lemma 3.3 we see that a connected component of $M^{T_2}$ is of type $(6,2,0)$ or $(4,4,0)$. If some component has type $(6,2,0)$ then the Euler characteristic of $M$ is equal to 2 by Lemma 3.5 since $\chi(M)$ is even.

So assume all components of $M^{T_2}$ are of type $(4,4,0)$. Let $\sigma_1 \in T$ be a non-trivial involution different from $\sigma$ and let $F_1$ denote the unique 4-dimensional connected component of $M^{\sigma_1}$. Note that $M^{T_2} \subset F_1$. By

Theorem 2.3 $\chi(F_1) \leq 3$. Since $\chi(M)$ is even we get $\chi(M) = \chi(M^{T_2}) = \chi(F_1) \leq 2$. Now the lemma follows from Theorem 1.3. ∎

**Proof of Theorem 1.1:** By the lemmas above we may assume that $\dim M^\sigma = 4$ for every non-trivial involution $\sigma \in T$. In view of Lemma 3.3 every connected $T_2$-fixed point component $X$ is of type $(4,4,0)$, $(4,2,2)$ or $(4,4,4)$. In the first two cases $X$ is an isolated fixed point whereas in the third case an inspection of the isotropy representation shows that $X$ is of dimension two.

Let $\sigma_i \in T$, $i = 1,2,3$, denote the non-trivial involutions and let $F_i$ denote the unique 4-dimensional connected component of $M^{\sigma_i}$. By the intersection theorem any two of the $F_i$'s intersect. Note that $T$ acts non-trivially on $F_i$ and, hence, $\chi(F_i) \leq 3$ by Theorem 2.3. If $\chi(F_i) = 3$ for some $i$, i.e. if $F_i$ is homeomorphic to $\mathbb{C}P^2$, then $F_i$ contains a $T_2$-fixed point component of positive dimension (cf. Theorem 2.5) which is necessarily of type $(4,4,4)$. Hence, if none of the $T_2$-fixed point components is of type $(4,4,4)$ then $\chi(F_i) \leq 2$ for all $i$ and

$$\chi(M) = \sum_i \chi(F_i) - \sum_{i<j} \chi(F_i \cap F_j) \leq 3 \cdot 2 - 3 = 3.$$

Since $\chi(M) \geq 2$ by Theorem 1.3 we are done in this case.

In the other case the intersection of the $F_i$'s contains a 2-dimensional $T_2$-fixed point component $X$ of type $(4,4,4)$. It follows from Lemma 3.2 and Theorem 2.5 that $X$ is diffeomorphic to $\mathbb{C}P^1$. Hence, $\chi(M) \leq 3 \cdot 3 - 2 \cdot 2 = 5$, and equality holds if and only if each $F_i$ is homeomorphic to $\mathbb{C}P^2$. Moreover in the equality case the $T_2$-fixed point components different from $X$ are all of type $(4,2,2)$ and for each $\sigma_i$ the fixed point manifold $M^{\sigma_i}$ is the union of $F_i$ and a 2-dimensional sphere.

We claim that $\chi(M) \neq 4$. Suppose to the contrary that $\chi(M) = 4$. Then we may assume that at least one of the $F_i$'s, say $F_1$, has Euler characteristic equal to 3. Now $\chi(M) = 4$ implies that the fixed point manifold $M^{\sigma_1}$ is the union of $F_1$ and an isolated fixed point $q$ (in fact, arguing as for $X$ we see that any $\sigma_1$-fixed point component of positive dimension different from $F_1$ would be diffeomorphic to $\mathbb{C}P^1$ implying $\chi(M) > 4$). Note that $q$, being an isolated $\sigma_1$-fixed point, must be of type $(4,4,0)$. Hence, $q$ belongs to $F_2$ and $F_3$. This implies $\chi(F_2) = \chi(F_3) = 3$.

On the other hand $F_1^{T_2}$ is the union of $X$ and a point $q'$, different from $q$, which is of type $(4,4,0)$ or $(4,2,2)$. If $q'$ has type $(4,4,0)$ then $\chi(F_2) \geq 4$ or $\chi(F_3) \geq 4$ which contradicts $\chi(F_2) = \chi(F_3) = 3$. If $q'$ has type $(4,2,2)$ then $\chi(M^{\sigma_2}) \geq 5$ and $\chi(M^{\sigma_3}) \geq 5$ which contradicts $\chi(M) = 4$.

Hence, $\chi(M) \neq 4$. Since $\chi(M) \leq 5$ and $\chi(M) \geq 2$ (by Theorem 1.3) we get $\chi(M) = 2, 3, 5$. ∎

Recall that for an unorientable even dimensional manifold of positive curvature a two-fold cover (the orientation cover) is simply connected [55]. Hence, Theorem 1.1 implies

**Corollary 3.9.** *Let $M$ be an unorientable 8-dimensional manifold. If $M$ admits a metric of positive curvature and symmetry rank $\geq 2$ then $\chi(M) = 1$.* ∎

# 4 Rationally elliptic manifolds

In this section we apply Theorem 1.1 to rationally elliptic manifolds. Recall that a closed simply connected $n$-dimensional manifold $M$ is *rationally elliptic* if its rational homotopy $\pi_*(M) \otimes \mathbb{Q}$ is of finite rank. Rational ellipticity imposes strong topological constrains. For example, Halperin has shown that the Euler characteristic of a rationally elliptic manifold is non-negative and that all odd Betti numbers vanish if the Euler characteristic is positive (cf. [28, Th. 1′, p. 175]). One also knows that the sum of degrees of generators of $\pi_{2*}(M) \otimes \mathbb{Q} := \bigoplus_i \pi_{2i}(M) \otimes \mathbb{Q}$ is $\leq n$ by work of Friedlander and Halperin (cf. [20, Cor. 1.3]).

On the other hand the class of rationally elliptic manifolds contains some interesting families, e.g. Lie groups, homogeneous spaces, biquotients and manifolds of cohomogeneity one. Moreover all simply connected manifolds presently known to admit a metric of non-negative curvature are rationally elliptic.

Using the information on the Euler characteristic given in Theorem 1.1 we obtain the following classification result for rationally elliptic manifolds. I like to thank Mikiya Masuda [41] for explaining to me properties of $\mathbb{Z}/2\mathbb{Z}$-cohomology $\mathbb{C}P^4$'s with $T$-action which are used in the proof.

**Theorem 4.1.** [Theorem 1.2] *Let $M$ be a simply connected positively curved 8-dimensional manifold of symmetry rank $\geq 2$.*

1. *If $M$ is rationally elliptic then $M$ has the rational cohomology ring and the rational homotopy type of a rank one symmetric space, i.e. of $S^8$, $\mathbb{H}P^2$ or $\mathbb{C}P^4$.*
2. *If $M$ is rationally elliptic and $H^*(M;\mathbb{Z})$ is torsion-free then $M$ is homeomorphic to $S^8$, diffeomorphic to $\mathbb{H}P^2$ or tangentially equivalent to $\mathbb{C}P^4$.*

**Proof:** Ad 1: We will show that $M$ has the rational cohomology ring of $S^8$, $\mathbb{H}P^2$ or $\mathbb{C}P^4$. From this one easily deduces that $M$ is formal. Hence, $M$ has the same rational homotopy type as $S^8$, $\mathbb{H}P^2$ or $\mathbb{C}P^4$.

By Theorem 1.1 the Euler characteristic of $M$ is $2, 3$ or $5$. Since $M$ is rationally elliptic the rational cohomology ring of $M$ is concentrated in even degrees (cf. [28, Th. 1′, p. 175]). If $\chi(M) = 2$ or $\chi(M) = 3$ then $M$ has the rational cohomology ring of $S^8$ or $\mathbb{H}P^2$, respectively. This follows directly from Poincaré duality. If $\chi(M) = 5$ then the rational cohomology ring of $M$ belongs to one of the following three cases:

1. $b_2(M) = 0$, $b_4(M) = 3$.
2. $b_2(M) = b_4(M) = 1$ and $x^2 = 0$ for a generator $x$ of $H^2(M;\mathbb{Q})$.
3. $b_2(M) = b_4(M) = 1$ and $x^2 \neq 0$ for a generator $x$ of $H^2(M;\mathbb{Q})$.

According to Friedlander and Halperin (cf. [20, Cor. 1.3]) the sum of degrees of generators of $\pi_{2*}(M) \otimes \mathbb{Q}$ is $\leq 8$. This excludes the first two cases. In fact, in the first case the minimal model of $M$ must have three generators of degree 4 and in the second case the minimal model of $M$ must have generators of degree $2, 4$ and $6$. So only the third case can occur, i.e. $b_2(M) = b_4(M) = 1$ and $x^2 \neq 0$ for a generator $x$ of $H^2(M;\mathbb{Q})$. By Poincaré duality $M$ has the rational cohomology ring of $\mathbb{C}P^4$.

Ad 2: Now assume $M$ is rationally elliptic and $H^*(M;\mathbb{Z})$ is torsion-free. If $\chi(M) = 2$ then $M$ is rationally a sphere by the first part. Since $H^*(M;\mathbb{Z})$ is torsion-free $M$ is an integral cohomology $S^8$ in this case. Being simply connected $M$ is a homotopy sphere and, hence, homeomorphic to $S^8$ by the work of Smale [52].

If $\chi(M) = 3$ then $M$ is an integral cohomology $\mathbb{H}P^2$ by Poincaré duality. We fix the orientation of $M$ for which the signature of $M$ is one. Note that $M$ is 3-connected and, hence, $M$ is a *Spin*-manifold. By the Atiyah-Hirzebruch vanishing theorem (see Theorem 2.4, Part 5) the $\hat{A}$-genus of $M$ vanishes (this follows also from Lichnerowicz' result [39] on the vanishing of the $\hat{A}$-genus for *Spin*-manifolds with positive scalar curvature). Since in dimension eight the space of Pontrjagin numbers is spanned by the $\hat{A}$-genus and the signature the manifolds $M$ and $\mathbb{H}P^2$ have the same Pontrjagin numbers.

From the work of Smale (cf. [53, Th. 6.3)] follows that $M$ admits a Morse function with three critical points. The classification results of Eells and Kuiper for these manifolds (cf. [12, Th. on p. 216]) imply that the diffeomorphism type of $M$ is determined by its Pontrjagin numbers up to connected sums with homotopy spheres. In particular, $M$ and $\mathbb{H}P^2$ are homeomorphic and diffeomorphic up to connected sum with a homotopy sphere. Recently Kramer and Stolz used Kreck's surgery theory to show that the action of the group of homotopy spheres on $\mathbb{H}P^2$ via connected sum is trivial (cf. [38, Th. A]). Hence, $M$ and $\mathbb{H}P^2$ are diffeomorphic.

Finally we consider the case $\chi(M) = 5$. From the first part we know already that $M$ is a rational cohomology $\mathbb{C}P^4$. Below we will show that $M$ is in fact an integral cohomology $\mathbb{C}P^4$. Assuming this for the moment we now prove that $M$ is tangentially equivalent to $\mathbb{C}P^4$, i.e. there exists a homotopy equivalence $f : M \to \mathbb{C}P^4$ such that $f^*(T\mathbb{C}P^4)$ and $TM$ are stably isomorphic.

We first note that $M$ and $\mathbb{C}P^4$ are homotopy equivalent since $M$ is an integral cohomology $\mathbb{C}P^4$ and simply connected. In the early 1970s Petrie conjectured that a smooth $S^1$-manifold $N$ which is homotopy equivalent to $\mathbb{C}P^n$ has the same Pontrjagin classes as $\mathbb{C}P^n$, i.e. the total Pontrjagin class $p(\mathbb{C}P^n)$ is mapped to $p(N)$ under a homotopy equivalence $N \to \mathbb{C}P^n$. Petrie's conjecture holds for $n = 4$ (cf. [33]). Hence, the homotopy equivalence $f : M \to \mathbb{C}P^4$ maps the Pontrjagin classes of $\mathbb{C}P^4$ to $M$. It is known that in this situation $M$ and $\mathbb{C}P^4$ are tangentially equivalent (cf. [48, p. 140]). For the convenience of the reader we sketch the argument: Since $H^*(M;\mathbb{Z})$ is torsion-free the condition on the Pontrjagin classes implies that the complexified vector bundles $TM \otimes \mathbb{C}$ and $f^*(T\mathbb{C}P^4) \otimes \mathbb{C}$ agree in complex $K$-theory. For $M$ a homotopy complex projective space of complex dimension $\not\equiv 1 \bmod 4$ the complexification map $KO(M) \to K(M)$ is injective. Hence, the real vector bundles $TM$ and $f(T\mathbb{C}P^4)$ are stably isomorphic (in fact, they are isomorphic since they have up to sign the same Euler class).

To complete the proof we need to show that $M$ is an integral cohomology $\mathbb{C}P^4$. We fix the orientation on $M$ for which the signature of $M$ is 1. Since $H^*(M;\mathbb{Z})$ is torsion-free it follows from Poincaré duality that $M$ is a twisted $\mathbb{C}P^4$, i.e. there are generators $x_{2i} \in H^{2i}(M;\mathbb{Z})$, $i = 1, 2, 3, 4$, and an integer $m > 0$, such that $x_8$ is the preferred generator with respect

to the chosen orientation and

$$x_2 \cdot x_6 = x_4^2 = x_8, \quad x_2^2 = m \cdot x_4, \quad x_2 \cdot x_4 = m \cdot x_6.$$

Let $T$ denote the two-dimensional torus which acts isometrically and effectively on $M$, let $T_2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ denote the 2-torus in $T$ and let $\sigma \in T_2$ be a non-trivial involution. By Theorem 2.4, Part 4, the codimension of $M^\sigma$ is 2 or 4. If the codimension of $M^\sigma$ is 2 then $M$ is an integral cohomology $\mathbb{C}P^4$, i.e. $m = 1$. This follows directly from the proof of Corollary 2.1.

So we are left with the case that $\dim M^\sigma = 4$ for every non-trivial involution $\sigma \in T$. Let $\sigma_i \in T$, $i = 1, 2, 3$, denote the non-trivial involutions. From the discussion in the previous section (see the proof of Theorem 1.1) we recall the following facts. For each $\sigma_i$ the fixed point manifold $M^{\sigma_i}$ is the union of a 4-dimensional connected component $F_i$ and a 2-dimensional sphere $S_i^2$. Moreover, $F_i$ is homeomorphic to $\mathbb{C}P^2$, the three $F_i$'s intersect in a 2-dimension $T_2$-fixed point component $X$ of type $(4, 4, 4)$ and $X$ is diffeomorphic to $\mathbb{C}P^1$. We fix the orientation on $F_i$ for which the signature of $F_i$ is 1.

It follows that the normal bundle of $X$ in $M$ is isomorphic as a real vector bundle to three copies of the Hopf bundle. In particular, the normal bundle is not spin and the restriction of the second Stiefel-Whitney class of $M$ to $X$ is non-zero. This shows that the restriction homomorphism ${f_i}^* : H^2(M;\mathbb{Z}) \to H^2(F_i;\mathbb{Z})$ induced by the inclusion $f_i : F_i \hookrightarrow M$ maps $x_2$ to an odd multiple of a generator $x$ of $H^2(F_i;\mathbb{Z}) \cong H^2(\mathbb{C}P^2;\mathbb{Z})$, i.e. ${f_i}^*(x_2) = a \cdot x$, $a$ odd. By applying the Lefschetz fixed point formula for the equivariant signature (cf. [4]) and Theorem 2.4, Part 4, it follows that the Euler class $e(\nu_i)$ of the normal bundle $\nu_i$ of $F_i \hookrightarrow M$ is equal to the preferred generator $x^2 \in H^4(F_i;\mathbb{Z})$. Hence, ${f_i}^*(x_4) = e(\nu_i) = x^2$. By restricting the identity $x_2^2 = m \cdot x_4$ to $F_i$ we see that $m = a^2$ is an odd square. In particular, $M$ is a $\mathbb{Z}/2\mathbb{Z}$-cohomology $\mathbb{C}P^4$.

Next we recall from the proof of Theorem 1.1 that $M^{T_2}$ is the union of $X$ and three points $p_i$, $i = 1, 2, 3$, with $p_i \in F_i$. We fix a lift $\xi \in H^2_T(M;\mathbb{Z})$ of $x_2$ and denote by $w_i$ the restriction of $\xi$ to $p_i$. By the structure theorem (cf. [31, Th. (VI.I), p. 106]) for rational cohomology complex projective spaces the kernel of the restriction homomorphism $H^*_T(M;\mathbb{Q}) \to H^*_T(p_i;\mathbb{Q})$ is generated by $(\xi - w_i)$.

The following argument which is due to Masuda shows that $m$ is equal to 1. Let $f_{i!} : H^*_T(F_i;\mathbb{Z}) \to H^{*+4}_T(M;\mathbb{Z})$ denote the equivariant Gysin map (or push-forward) induced by $f_i : F_i \hookrightarrow M$. For properties of the Gysin map see for example [40, p. 132-133].

*Claim 1*: $f_{i!}(1) = \frac{1}{a^2}(\xi - w_j) \cdot (\xi - w_k)$ where $\{i, j, k\} = \{1, 2, 3\}$.

*Proof:* Since $p_j$ and $p_k$ are not in $F_i$ the restriction of $f_{i!}(1)$ to each of these points must vanish. Hence, $f_{i!}(1)$ is divisible by $(\xi - w_j) \cdot (\xi - w_k)$. Comparing degrees we find that $f_{i!}(1) = c \cdot (\xi - w_j) \cdot (\xi - w_k)$ for some rational constant $c$. By restricting this identity to ordinary cohomology we obtain $x_4 = c \cdot x_2^2$, and, hence, $c = \frac{1}{m} = \frac{1}{a^2}$. ✓

*Claim 2*: $w_i - w_j$ is divisible by $a^2$ in $H^2(BT;\mathbb{Z})$.

*Proof:* From the first claim we deduce

$$f_{i!}(1) - f_{j!}(1) = \frac{1}{a^2}\left((w_i - w_j)\cdot \xi - (w_i - w_j)\cdot w_k\right).$$

Since $(1, \xi)$ is part of a basis of the free $H^*(BT;\mathbb{Z})$-module $H^*_T(M;\mathbb{Z})$ it follows that $(w_i - w_j)$ is divisible by $a^2$. ✓

Recall that any two $T$-fixed points $p_i, p_j$, $i \neq j$, are contained in a $T$-invariant 2-dimensional sphere $S^2_k$ which is fixed pointwise by the involution $\sigma_k$ (where $\{i, j, k\} = \{1, 2, 3\}$). Consider the T-action on $S^2_k$ and let $m_{ij} \in H^2(BT;\mathbb{Z})$ denote the weight of the tangential $T$-representation at $p_i$. Note that $m_{ij}$ is only defined up to sign and $m_{ij} = \pm m_{ji}$ (here and in the following the notation $\alpha = \pm\beta$ is a shortcut for $\alpha = \beta$ or $\alpha = -\beta$).

*Claim 3*: $\pm a \cdot m_{ij} = w_i - w_j$ and $m_{ij}$ is divisible by $a$.

*Proof:* First note that the normal bundle of $F_i$ in $M$ restricted to $p_i$ has weights $\pm m_{ij}, \pm m_{ik}$, where $\{i, j, k\} = \{1, 2, 3\}$. Hence, by restricting the identity given in the first claim to the $p_i$'s we obtain the following identities in the polynomial ring $H^*(BT;\mathbb{Z})$:

$$\pm a^2 \cdot m_{12} \cdot m_{13} = (w_1 - w_2)\cdot(w_1 - w_3)$$

$$\pm a^2 \cdot m_{23} \cdot m_{21} = (w_2 - w_3)\cdot(w_2 - w_1)$$

$$\pm a^2 \cdot m_{31} \cdot m_{32} = (w_3 - w_1)\cdot(w_3 - w_2)$$

Since $m_{ij}$ and $m_{ji}$ agree up to sign $\pm a \cdot m_{ij} = w_i - w_j$ and, using Claim 2, $m_{ij}$ is divisible by $a$. ✓

Suppose $X \subset M^T$. Then we can choose a subgroup $S^1$ of $T$ such that $F_1$ is fixed pointwise by $S^1$. By Claim 3 the subgroup $\mathbb{Z}/a\mathbb{Z}$ of $S^1$ acts trivially on $M$. Since the $T$-action is effective we get $a = \pm 1$.

If $X \not\subset M^T$ then $M^T$ consists of five isolated points $\{p, p', p_1, p_2, p_3\}$ where $p, p' \in X$ and $p_i \in F_i$. Recall that $F_i$ is homeomorphic to $\mathbb{C}P^2$. In particular, there is a unique $T$-invariant 2-dimensional sphere in $F_i$ which contains $p$ and $p_i$. Consider the $T$-action on this sphere and let $m_i \in H^2(BT;\mathbb{Z})$ denote the weight of the tangential $T$-representation at $p_i$. Similarly, let $m'_i \in H^2(BT;\mathbb{Z})$ denote the weight of the tangential $T$-representation which corresponds to $p'$ and $p_i$. Note that $m_i$ and $m'_i$ are only defined up to sign.

Let $w$ and $w'$ denote the restriction of $\xi$ to $p$ and $p'$, respectively. Since any torus action on a homotopy $\mathbb{C}P^2$ is of linear type and $x_2$ restricted to $F_i$ is equal to $a$ times a generator of $H^2(F_i;\mathbb{Z})$ we get

$$\pm a \cdot m_i = w_i - w \quad \text{and} \quad \pm a \cdot m'_i = w_i - w'. \tag{1}$$

Now consider the circle subgroup $S \hookrightarrow T$ defined by $w = w'$. Since $T$ acts linearly on $F_i$ the fixed point set $F_i^S$ is the union of $X$ and $p_i$.

For $u \in H^*(BT;\mathbb{Z})$ let $\bar{u}$ denote the restriction of $u \in H^*(BT;\mathbb{Z})$ to $H^*(BS;\mathbb{Z})$. Since $\bar{w} = \bar{w}'$ it follows from equations (1) that $\bar{m}_i$ and $\bar{m}'_i$

agree up to sign. Since $T$ acts effectively on $M$ and the weights $m_{ij}$ are divisible by $a$ (see Claim 3) $\bar{m}_i$ and $\bar{m}_i'$ are both coprime to $a$.

Suppose $m = a^2$ is not equal to 1. Consider the action of $\mathbb{Z}/a\mathbb{Z} \subset S$. Since $\bar{m}_{ij}$ is divisible by $a$ and $\bar{m}_i$ and $\bar{m}_i'$ are both coprime to $a$ the connected $\mathbb{Z}/a\mathbb{Z}$-fixed point component $Z$ which contains $p_1$ contains both $p_2$ and $p_3$ but does not contain $X$. Hence, the $S$-equivariant Gysin map $f_! : H_S^*(Z;\mathbb{Z}) \to H_S^{*+4}(M;\mathbb{Z})$ induced by the inclusion $f : Z \hookrightarrow M$ vanishes after restricting to $X$. Applying the structure theorem (cf. [31, Th. (VI.I), p. 106]) for rational cohomology complex projective spaces to $M$ and $Z$ we find that $f_!(1)$ is divisible by $(\bar{\xi} - \bar{w})^2$. Comparing degrees it follows that there is a rational constant $C$ such that $f_!(1) = C \cdot (\bar{\xi} - \bar{w})^2$. By restricting this identity to the $T$-fixed point $p_i$ we obtain $\pm \bar{m}_i \cdot \bar{m}_i' = C \cdot (\bar{w}_i - \bar{w})^2$. Using equations (1) we get $C = \pm \frac{1}{a^2}$. Hence, $\frac{1}{a^2} \cdot (\bar{\xi} - \bar{w})^2 \in H_S^4(M;\mathbb{Z})$. Recall from the first claim that $\frac{1}{a^2}(\bar{\xi} - \bar{w}_1) \cdot (\bar{\xi} - \bar{w}_2)$ is also in $H_S^4(M;\mathbb{Z})$. Taking the difference of these two elements, we obtain

$$\frac{1}{a^2}\left((\bar{w}_2 - \bar{w}_1 + 2 \cdot (\bar{w}_1 - \bar{w})) \cdot \bar{\xi} + (\bar{w}^2 - \bar{w}_1 \cdot \bar{w}_2)\right) \in H_S^4(M;\mathbb{Z}).$$

Since $(1, \bar{\xi})$ is part of a basis of the free $H^*(BS;\mathbb{Z})$-module $H_S^*(M;\mathbb{Z})$ it follows that $(\bar{w}_2 - \bar{w}_1 + 2 \cdot (\bar{w}_1 - \bar{w}))$ is divisible by $a^2$. Now $(\bar{w}_2 - \bar{w}_1)$ is divisible by $a^2$ by Claim 3 and $a$ is odd. Hence, $(\bar{w}_1 - \bar{w})$ is divisible by $a^2$. Using equations (1) we deduce that $a$ divides $\bar{m}_i$. This contradicts $a^2 \neq 1$ since $\bar{m}_i$ is coprime to $a$. Hence, $m = a^2$ is equal to 1. ■

We close this section with an application to biquotients. Recall that any biquotient of a compact connected Lie group $G$ is rationally elliptic and comes with a metric of non-negative curvature induced from a bi-invariant metric on $G$.

**Corollary 4.2.** *A simply connected 8-dimensional biquotient of positive curvature and symmetry rank $\geq 2$ is diffeomorphic to $S^8$, $\mathbb{C}P^4$, $\mathbb{H}P^2$ or $G_2/SO(4)$.*

**Proof:** According to Theorem 4.1 a simply connected positively curved 8-dimensional biquotient with symmetry rank $\geq 2$ is rationally singly generated. Rationally singly generated biquotients where classified by Kapovich and Ziller (see [35, Th. A]). In dimension 8 these are the homogeneous spaces given in the corollary. ■

**Remark 4.3.** From the classification of homogeneous positively curved manifolds follows that $G_2/SO(4)$ does not admit a homogeneous metric of positive curvature. We don't know whether $G_2/SO(4)$ admits a positively curved metric with symmetry rank two.

# 5 Bordism type

In this section we consider the bordism type of closed simply connected Riemannian 8-manifolds with positive curvature. We determine the *Spin*-bordism type and comment on the oriented bordism type for manifolds with symmetry rank $\geq 2$.

One knows that in dimension eight the *Spin*-bordism group $\Omega_8^{Spin}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and the *Spin*-bordism type is detected by Pontrjagin numbers (cf. [42, p. 201]). Since in this dimension the Pontrjagin numbers are uniquely determined by the $\hat{A}$-genus and the signature it suffices to compute these numerical invariants.

**Proposition 5.1.** *Let $M$ be an 8-dimensional Spin-manifold. If $M$ admits a metric of positive curvature and symmetry rank $\geq 2$ then $\chi(M) = 2$ or 3 and $M$ is Spin-bordant to $S^8$ or $\pm\mathbb{H}P^2$.*

**Proof:** By the Atiyah-Hirzebruch vanishing theorem (see Theorem 2.4, Part 5) or alternatively by Lichnerowicz' theorem [39] the $\hat{A}$-genus of $M$ vanishes. Hence, it suffices to show that the signature of $M$ is equal to the signature of $S^8$ or $\pm\mathbb{H}P^2$, i.e. we want to show that $|sign(M)| \leq 1$. If some isometry in $T$ acts with a fixed point component of codimension 2 this follows directly from Corollary 2.1 and Theorem 1.1 (in fact $M$ is bordant to $S^8$ in this case since an integral cohomology $\mathbb{C}P^4$ is never spin). So assume that for any $\tau \in T$ the fixed point manifold

$$M^\tau \text{ has no fixed point component of codimension 2.} \qquad (*)$$

Recall from Theorem 1.1 that $\chi(M) = 2, 3$ or 5. If $\chi(M) = 2$ then $sign(M) = 0$. To see this consider a subgroup $S^1 \subset T$ of positive fixed point dimension (i.e. $\dim M^{S^1} > 0$). By condition $(*)$ any connected component of $M^{S^1}$ is of dimension $\leq 4$. It follows from Theorem 2.4, Part 1, that $M^{S^1}$ is $S^2$ or an integral cohomology $S^4$. Since the signature of $M$ is the sum of the signatures of the connected components of $M^{S^1}$ the signature of $M$ vanishes (see Theorem 2.4, Part 3).

If $\chi(M) = 3$ then $|sign(M)| = 1$. The reasoning is similar to the one above. Choose an $S^1$-subgroup of $T$ such that the fixed point manifold $M^{S^1}$ has a connected component $F$ of dimension 2 or of dimension 4. Any such $F$ is simply connected by [55] and satisfies $|sign(F)| \leq \chi(F) - 2$. Since $\chi(M) = \chi(M^{S^1}) = 3$ and the signature of $M$ is the sum of the signatures of the connected components of $M^{S^1}$ (taken with the appropriate orientation) we get $|sign(M)| = 1$.

Finally we claim that the case $\chi(M) = 5$ cannot occur. First note that in this case the signature of $M$ is odd since $sign(M) \equiv \chi(M) \bmod 2$. Let $\sigma_i$, $i \in \{1, 2, 3\}$, denote the three non-trivial involutions in $T$. It follows from condition $(*)$ and Theorem 2.4, Part 4, that $M^{\sigma_i}$ contains a 4-dimensional connected component $F_i$ (which is unique by the intersection theorem). By Lemma 3.2 $F_i$ is homeomorphic to $S^4$ or $\mathbb{C}P^2$. Since $M$ is spin the action of $\sigma_i$ must be even (see Theorem 2.4, Part 6). Hence, $M^{\sigma_i}$ is the union of $F_i$ and isolated $\sigma_i$-fixed points. Using Lemma 3.3 we see that each connected component of $M^{T_2}$ has type $(4, 4, 4)$ or $(4, 4, 0)$. In particular, any $T_2$-fixed point component is contained in some $F_i$.

To derive a contradiction we will compute the Euler characteristic. Consider the case that for one of the $F_i$'s, say $F_1$, the Euler characteristic is equal to 3 and, hence, $F_i$ is homeomorphic to $\mathbb{C}P^2$. Since $\sigma_2$ acts non-trivially on $F_1$ we get $F_1^{\sigma_2} = F_1^{\sigma_3} = S^2 \cup \{pt\}$, where $S^2$ and $pt$ are connected components of $M^{T_2}$ of type $(4, 4, 4)$ and $(4, 4, 0)$, respectively.

Hence, one of the other $F_i$'s, say $F_2$, contains $S^2 \cup \{pt\}$. This leads to the contradiction

$$5 = \chi(M) = \chi(M^{T_2}) = \chi(F_2) + \chi(F_3) - \chi(F_2 \cap F_3) \leq 3 + 3 - 2 = 4.$$

So suppose $\chi(F_i) = 2$ for all $i$. Note that $M^{T_2}$ cannot contain a connected component of type $(4,4,4)$ since otherwise $\chi(M) = 2$ by a computation similar to the one above. Hence, each connected component of $M^{T_2}$ is of type $(4,4,0)$. In particular, the $F_i$'s intersect pairwise in different points which gives the contradiction

$$5 = \chi(M) = \sum_i \chi(F_i) - \sum_{i<j} \chi(F_i \cap F_j) + \chi(F_1 \cap F_2 \cap F_3) \leq 6 - 3 = 3.$$

In summary, we have shown that $\chi(M) \neq 5$. This completes the proof of the theorem. ■

A more natural and apparently more difficult problem is to understand the *oriented* bordism type of an 8-dimensional positively curved manifold $M$ with symmetry rank $\geq 2$.

Wall has shown that the only torsion in the oriented bordism ring $\Omega_*^{SO}$ is 2-torsion and the oriented bordism type of a manifold is determined by Pontrjagin- and Stiefel-Whitney numbers [57]. In dimension eight $\Omega_8^{SO}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Hence, in this dimension the oriented bordism type of an oriented manifold is determined by its Pontrjagin numbers.

In contrast to the case of *Spin*-manifolds the $\hat{A}$-genus does not have to vanish on positively curved 8-dimensional oriented manifolds with symmetry (consider for example $\mathbb{C}P^4$). This makes the problem of determining the oriented bordism type more difficult. One way to attack this problem is to show the stronger statement that for some orientation of $M$ and some $S^1 \subset T$ the $S^1$-action has locally the same $S^1$-geometry as a suitable chosen $S^1$-action on one of the symmetric spaces $S^8$, $\mathbb{H}P^2$ or $\mathbb{C}P^4$ (that two $S^1$-manifolds have the same local $S^1$-geometry just means that there exists an equivariant orientation preserving diffeomorphism between the normal bundles of the $S^1$-fixed point manifolds). Once this has been accomplished one can glue the complements of the normal bundles together to get a new manifold $W$ with fixed point free $S^1$-action which is bordant to the difference of $M$ and the symmetric space in question. As observed by Bott [5] all Pontrjagin numbers of a manifold with fixed point free $S^1$-action vanish and, hence, $W$ is rationally zero bordant. Since the oriented bordism ring has no torsion in degree 8 the manifold $M$ is bordant to the symmetric space in question.

This line of attack can be applied successfully at least if $\chi(M) \neq 5$. Details will appear elsewhere.

It is interesting to compare the results above with [10] in which it is shown that there exists an infinite sequence of closed simply connected Riemannian 8-manifolds with *nonnegative* curvature and mutually distinct oriented bordism type.

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